

THE TEACHING OF THE CONCEPT OF TANGENT LINE USING ORIGINAL SOURCES

Catarina Mota^a, Maria Elfrida Ralha^b, Maria Fernanda Estrada^c

^aCMAT & Didáxis – Cooperativa de Ensino

^bCMAT & Departamento de Matemática e Aplicações, Universidade do Minho

^cCMAT – Centro de Matemática, Universidade do Minho

This paper reports a mathematics experiment conducted in a Portuguese secondary school, with 11th and 12th grades pupils (pre-university). We have used original historical sources for teaching the mathematical concept of tangent line to a curve as well as its connection with the concept of derivative. Emphasis will be given to the conception of the designed module, to the analysis of the pupils' answers and to the pupils' final outcomes. We will infer both benefits and disadvantages of using such activities into the teaching of mathematics.

Keywords: Tangent line; derivative; historical sources; history of mathematics; mathematics education.

INTRODUCTION

The use of history of mathematics in mathematics teaching is increasingly advocated by experts in mathematics, history of mathematics and mathematics education. Many reasons have been presented to justify why it is advantageous to use history of mathematics in mathematics teaching. As reported by several authors (Fauvel&Maanen, 2000; Fried, 2001; Jankvist, 2009) history of mathematics can be used to humanize mathematics, as a motivational tool, to present a dynamical vision of mathematics and its evolution and/or to support the development of an internal image of mathematical concepts, etc...

Despite the advantages, about the use of history of mathematics in mathematics teaching, some disadvantages have also been identified. A list of sixteen reasons for not using history of mathematics in classrooms is, for example, presented by Siu (2006, p. 268-269) from which we emphasise the time consuming problem, the pupils' concerns about the benefits of this use and the lack of teacher training.

In Portugal there is a tradition in the use of history of mathematics in mathematics teaching (Mota & Ralha, 2011), which dates back to 1772 when a Faculty of Mathematics was created at the newly reformed University of Coimbra. The importance of the history of mathematics in Portugal is also confirmed by the numerous works on the history of Portuguese mathematics and mathematicians, namely *Ensaio histórico da origem e dos progressos das mathematicas em Portugal*, the 1st ever history of mathematics of a country, published in 1819 by Francisco de Borja Garção-Stokler.

Nowadays we still have proofs on the importance of the history of mathematics, by having all mathematics teachers' public university degrees in Portugal attending, at least, one semester course of History of Mathematics in their curricula.

According to this long lasting tradition we developed an experiment, as part of our PhD research on the historical evolution of the concept of tangent line and its teaching, which aims at examining the use of original historical sources in teaching mathematical concepts.

THE TEACHING OF THE CONCEPT OF TANGENT LINE

The concept of tangent line to a curve is a concept that becomes familiar to most pupils during their compulsory education and it is, therefore, expected that such concept doesn't present doubts neither to teachers nor to pupils. However, as Vinner showed (1991, p. 65-81), the pupils' concept definition and concept image of tangent line are not usually the same and this does not allow learners to correctly work with the concept. To avoid this situation, Tall (1990, p. 58) defended that students should face enriching learning experiences to help them construct concept images that are as close as possible to the concept definition.

On the other hand, history of mathematics may provide enriching learning experiences regarding the concept of tangent line, because this concept has been studied since the ancient Greece, in the works of Euclid, Archimedes and Apollonius, with an important development in the 17th century with the work of Roberval, Descartes, Fermat, Newton and Leibniz. These mathematicians, as well as the Portuguese mathematician José Anastácio da Cunha in the 18th century, had different approaches to the concept of tangent line, depending on the nature of their work [1].

As Tzanakis & Thomaidis (2011, p.1654-1655) showed history of mathematics can be used in mathematics teaching to, among others, uncover/unveil concepts, as a bridge between mathematics and other disciplines, to get insights into concepts by looking from a different point of view and to compare old and modern. We have tried to achieve these goals by using original sources in the teaching of the concept of tangent line.

The concept of tangent line in the Portuguese curriculum and textbooks

In Portugal, the concept of tangent line to a curve is a part of mathematics curriculum for the 11th and 12th grades (pupils aged 16 and 17) when studying the derivatives. Within these two years the concept of tangent line appears as a complement to the analytical approach of the concept of derivative, in the relationship between the derivative of a function at a given point and the slope of the tangent line to the graph of the function at that point (Departamento Ensino Secundário [DES], 2002, p. 6-7).

Despite the disadvantages mentioned in the introduction, the Portuguese curriculum for mathematics advocates the use of the history of mathematics in mathematics education, in particular in the introduction to differential calculus (and hence the concept of tangent line).

The use of historical examples or references to the evolution of mathematical concepts will help pupils to appreciate the contribution of mathematics to the problem understanding and problem solving by Man through time. Some suggested situations: Pedro Nunes' polynomials, history of differential calculus, history of complex numbers. (DES, 2001, p. 20)

Since the curriculum is vague in terms of how the history of differential calculus should be used, most textbooks and teachers use only the so called "illumination approaches", (Jankvist's categorization; 2009, p. 245-246). This approach aims at both informative and formative purposes, but most Portuguese textbooks [2] contain only a few images of the referenced mathematicians (even if there is no certainty that the image is a real picture of the person) and very scarce references (sometimes even misleading) to the work of these mathematicians. These notes often become more distracting than informative, not integrating the explanations of the contents and considered, by most of the teachers, only as curiosities that have no use in learning of the concepts.

Methodology and data collection

Our experiment took place in two different moments with different approaches. In a first phase, involving an 11th grade class of twenty-one pupils, we have used "the modules approach" (Jankvist, 2009, p. 246) with a "historical package" that had the duration of three-class periods of ninety minutes each. This choice was due to the fact that we wanted to focus on a small topic, with strong ties to the curriculum, and we had limited class periods available. The structure of the historical package is presented below.

Table 1: Structure of the historical package.

Author	Original Source used [3]	Question/task presented to pupils
Euclid	<i>Elements III</i> , Definition 2 (Heath, 1956, p. 1) and <i>Elements III</i> , 16 (corollary) (Heath, 1956, p. 39)	<ol style="list-style-type: none"> 1. Is this definition suitable to all curves? Justify. 2. Present a method to draw the tangent line to a circle at a given point.
Archimedes	<i>On Spirals</i> , Definition 1 (Heath, 2002, p. 165)	Is Euclid's definition suitable for Archimedes' spiral? In case of a negative answer present an alternative definition for a tangent line to a curve.
Roberval	<i>Observationes sur la composition des mouvements et sur le moyen de trouve les touschants de lignes</i>	<ol style="list-style-type: none"> 1. Use Roberval's method to draw the tangent line to a cycloid at a given point. Make a small report about the taken steps and justify it with transcriptions from the original text.

	<p><i>curves</i> (axiom or invention principle and general rule) (Roberval, 1736, p. 22-23)</p> <p><i>Traité des indivisibles</i> (explication of the Cycloid) (Roberval, 1736, p. 209-212)</p>	<p>2. Using the concepts presented by Roberval and your knowledge in the theory of movement establish connections between the concepts of movement, velocity and tangent line.</p>
Fermat	<p><i>Methodus ad Disquirendam Maximam et Minimam</i> (Fermat, 1891, p. 121)</p>	<p>1. Using Fermat's method, determine the extremes of a 2nd degree polynomial function.</p> <p>2. What is, geometrically, the interpretation of Fermat's method?</p>
Leibniz	<p><i>Nova methodus pro maximis et minimis, itemque tangentibus, qua nec irrationales moratur</i> (Leibniz, 1983, p. 5-7)</p>	<p>1. Using an image of the differential triangle, establish a relationship between the slope of a tangent line to a curve at a given point and the function derivative at that point.</p> <p>2. Establish a relationship between function monotony and the sign of the derivative.</p> <p>3. Using the established relationship present a method to determine the extreme of a function.</p>

The authors and tasks were chosen to represent the different approaches to the concept of tangent line as well as its relation with other concepts (the Greek “touch without cut” approach, the relationship with physics and, in particular, with movement, the limit position of the secant approach, the relationship with the concept of derivative, etc.).

Our pupils worked in five small heterogeneous groups but their answers, in the different groups, were quite similar. Hereby we present the most relevant data obtained from the students' written answers.

Table 2: Data collection of the module-based approaches.

Question/task	Collected Data
Euclid 1	All groups answered “no” and presented a graphic of a curve where the tangent line at a point cuts the curve at another point to justify their answer (all examples were similar to the graphic of a 3 rd degree polynomial function).
Euclid 2	All groups presented the correct method to draw a tangent line to a circle.

Archimedes	All groups answered “no” and presented an alternative definition that had as common idea the fact that the tangent line touches the curve at the contact point and no other in a neighbourhood.
Roberval	The pupils presented many difficulties in the implementation of the proposed tasks, having just managed to perform the task after a few explanations of the teacher, mainly one related to the description of curves through movement.
Fermat 1	All groups, after some discussion and still presenting some doubts about the method, were able to answer correctly to the question.
Fermat 2	Only one group was able to completely understand the geometrical method that had to be presented by the teacher.
Leibniz	All groups answered the questions correctly.

The second phase of the experiment took place almost one year later and involved pupils of the 12th grade, ten of them (group A) that had participated in the first part of the experiment and twelve (group B) that were participating for the first time in the experiment. In this phase, it was used “the history-based approaches” (Jankvist, 2009, p. 246-247) by confronting the pupils with different definitions of the concept of tangent line and propositions about this concept, presented in a historical order, that were discussed and analysed with the pupils in order to ~~develop~~ achieve a clearer and more complete concept image and concept definition of tangent line. We have chosen to work with the ~~authors~~ same mathematicians used in the first phase of the experiment, adding a new one: the Portuguese mathematician José Anastácio da Cunha (1744-1787) aiming at dealing with yet ~~who adds~~ a different point of view on the concept along with a national component. The definitions analyzed are the following:

Table 3: Structure of the history-based approaches.

Author	Definition / Proposition
Euclid	<i>Elements III</i> , Definition 2: A straight line is said to touch a circle which, meeting the circle and being produced, does not cut the circle. (Heath, 1956, p. 1)
Archimedes	<i>On spirals</i> , 13: If a straight line touch the spiral, it will touch it in one point only. (Heath, 2002, p. 167)
Roberval	<i>Observationes sur la composition des mouvements et sur le moyen de trouve les touschants de lignes curves</i> , Axiom or Invention Principle: The direction of the movement of one point which describes a curve is the tangent line of the curve at the position that the point occupies. (Roberval, 1736, p. 22)

Fermat [4]	Tangent line is the limit position of the secant when the two points of the curve tend to meet. (Eves, 1992, p. 391)
Leibniz	<i>Oeuvre concernant le Calcul Infinitésimal</i> : drawing the tangent line is to draw the straight line that joins two points on the curve that are distant from each other an infinitely small distance, or the side of a polygon of infinite angles that we consider equivalent to the curve. (Leibniz, 1983, p. 7)
José Anastácio da Cunha	<i>Princípios Matemáticos II</i> , Definition III: If the sides of an angle meet at the vertex such as it is not possible to draw two straight lines between them it is said that the sides are tangent to each other. (Cunha, 1790, p. 13)

The students were asked first to write their comments and opinions (the definition validity, evolution, applicability...), and afterwards to debate it in class sharing their opinions. The most receptive and participative pupils were the ones who took part in the first phase of the experiment and almost all comments, in the class debate, were made by those pupils. Hereby we present the most relevant data obtained from the students' written answers and from the field notes taken during the class debate.

Table 4: Data collection of the history-based approaches.

Author	Comments (group A)	Comments (group B)	Comments (class debate)
Euclid	This definition is not valid for all curves.	This definition is correct and is the one generally used.	Group A tried to convince group B that this definition shouldn't be used today since it is not valid for all curves by using examples of curves where the tangent line cuts the curve.
Archimedes	Reference to the uniqueness of the tangency point that is not valid if the curve is a straight line.	Accepted that this property is valid for all curves.	Group A presented their example to group B.
Roberval	Pupils were able to establish, even though with some difficulties, the connection with	Pupils weren't able to understand the relationship between the presented definition and their	An explanation on the part of the teacher was necessary in order to make the relationship between mathematics and physics evident to all

	physics.	concept image of tangent line.	pupils.
Fermat	Recognized this as the generally used definition of tangent line.	Established a relationship with the concept of derivate.	The two groups' approaches were considered complementary.
Leibniz	Pupils referred to the last part of the definition as having no connection with the rest and so it is unnecessary.	Established a relationship with the concept of derivate.	Recognized the similarity with the treatment given by Fermat and discarded the last part of the definition.
José Anastácio da Cunha	Stated the difficulty of the definition and weren't able to understand the connection of the definition with the geometrical treatment of the functions they are used to do.		The teacher explained the presented definition, presenting Anastácio da Cunha's use of the definition.

After analysing the presented definitions pupils were asked to present their own definition of tangent line. All of them used the Fermat's definition or a very similar one that they tried to write themselves.

CONCLUSIONS

The use of original sources to teach the concept of tangent line has proved to be not only important for us to realize many positive aspects, but also to identify some obstacles to overcome.

The first major difficulty was faced when constructing the historical package. The choice of the sources to be used, the translation of the original sources into Portuguese that have raised problems of terminology and notation, and the choice of the questions/tasks made the construction of the module to be time-consuming and scientifically intense. It is only possible to develop this type of work if the teacher has some teaching experience and a profound knowledge of the concept/subject to teach together with historical interest and practice. In our particular case, this work was only possible due to a previous investigation that we, ourselves, had conducted for our MSc degree. We are sure that we might never develop such competence if it was not for our own process of finding, disclosing and intense debate (with our supervisors) on many details on the studied sources. This calls our attention to the need of introducing/consolidating history of mathematics in teachers' training, in history of mathematics itself, in experiencing discoveries of historical material, on

debating doubts and results with specialists and on knowing to properly adapt such material in the teaching process. It also calls our attention to the need of a real collaborative work among colleagues that are mathematics teachers, to share materials, experiences, ideas that will make less time-consuming the preparation of this sort of activities.

During the three-class periods where the history package was used, it was hard to get the commitment and stimulate pupils' interest in this type of activity. The first arguments used by the pupils were that mathematics is not history, they do not attend history classes anymore and that they do not like history. Pupils were not, at all, aware of the importance of the historical development of mathematics for understanding mathematics itself, so they discard this sort of activities even before they begin working with them. To overcome this pupils' arguments/attitude history of mathematics might be introduced into mathematics teaching early and often and history classes and mathematics classes might even become, in some aspects, coordinated disciplines.

After the first refusal and the beginning of the task, pupils faced the complexity of the presented texts. The texts had been previously prepared by ourselves but, even so, since Portuguese pupils are not used to read mathematical texts, it was difficult for them to understand the presented historical material. This barrier might only be overcome if pupils are more often faced with reading and analysing mathematical texts.

Pupils also presented difficulties with the establishment of connections between mathematics and physics. Although Portuguese mathematics curriculum explicitly defends the interdisciplinarity of mathematics and physics, that is rarely a fact in teaching practices. This experiment shows us that a true collaborative work between the teachers of the two disciplines is necessary, for example, for pupils to realise the importance of mathematics to explain physic phenomena.

Another difficulty was related to the purpose of the task. In Portugal, history of mathematics is not a topic evaluated in final examinations, so pupils of these levels (prior to being selected for universities) tend to discard anything that will not be part of national examinations and to concentrate only in "model questions" which might prepare them to the final exam. Pupils showed only interest in the questions concerning to Leibniz's work, since it seemed to them that this might be useful for their exam preparation and they were not able to see any more immediate advantages in any other part of the work. However the real advantages for them with our work came to be fully acknowledged by the same pupils in the second year of the experiment. When asked to comment the definitions/propositions presented, the pupils who took part in the first phase of the experiment were the most receptive and participative ones. They also presented less doubts and less difficulties. This shows that, even though they had some initial concerns, the work made in the previous year developed the inwardness of concepts that gave them the knowledge to perform easily the 2nd phase tasks. At the end it was clear to all participants that the pupils

involved in the two years of the experiment had a more solid formation than the others, showing the utility of this sort of tasks to develop a profound and more complete knowledge of the concept. When faced with counterintuitive examples such as the tangent line to a straight line or the tangent line to a curve at an inflection point, pupils showed little difficulty and did not present the problems that are often reported (Vinner, 1991; Tall, 1990) concerning to the concept of tangent line.

It is our conviction that the contact with the historical evolution of the concept allowed our pupils to fully understand it: they internalized the concept, they constructed clearer concept images, they achieved a correct concept definition in their minds. They even performed better in their final examinations.

NOTES

1. For detailed information about the historical development of the concept of tangent line see Estrada, Mota & Ralha, 2011; Mota, 2008; Coolidge, 1951; Carvalho, 1919.
2. We analyzed the textbooks Matemática A 12º, Novo Espaço 12 and Xeqmat 12 which are the most used in the country.
3. All texts were translated into Portuguese by the authors.
4. Although Fermat does not present a definition of tangent line in his work, the treatment he gave to this concept clearly shows the definition he had in mind. Since this definition is historical important and is the one presented in Portuguese textbooks we choose to present it by using Eve's words, that we considered to be the most representative of Fermat's work.

REFERENCES

- CARVALHO, A. S. (1919). *A teoria das tangentes antes da invenção do cálculo diferencial*. Coimbra: Imprensa da Universidade.
- COOLIDGE, J. L. (1951). The story of the tangents. *The American Mathematical Monthly*, Vol. 58, N. ° 7, pp. 449-462.
- COSTA & RODRIGUES (2012). *Novo Espaço 12*. Porto: Porto Editora.
- Departamento Ensino Secundário (2001). *Programa de Matemática A 10º ano*. Lisboa: Ministério da Educação.
- Departamento Ensino Secundário (2002). *Programa de Matemática A 11º ano*. Lisboa: Ministério da Educação.
- ESTRADA, M. F., MOTA, C. & RALHA, M. E. (2011). The concept of tangent line: historical and didactical aspects in Portugal (18th century). In E. Barbin, M. Kronfeller & C. Tzanakis (Ed) *Proceedings of the Sixth European Summer University in History and Epistemology in Mathematics Education*, p. 687-695. Vienna: Verlag Holzhausen GmbH.
- EVES, H. (1992). *An Introduction to the History of Mathematics*. Orlando: Saunders College Publishing.

- FAUVEL J. & MAANEN J. V. (2000). *History in Mathematics Education: The ICMI Study*. Dordrecht. Kluwer Academic Publishers.
- FERMAT, P. (1891). *Oeuvres de Fermat*, Vol. 1. Paris: Gauthier-Villars et fils.
- FRIED, M. (2001). Can Mathematics Education and History of Mathematics Coexist? *Science & Education* 10, 391 – 408.
- GOMES et al (2012). *Xeqmat12*. Lisboa: Texto Editores.
- HEATH, T. (1956). *Euclid - The Thirteen books of The Elements*, Vol. 2. Dover Publications: New York.
- HEATH, T. (2002). *The works of Archimedes*. Dover Publications: New York.
- JANKVIST, U. T. (2009). A categorization of the ‘whys’ and ‘hows’ of using history in mathematics education. *Educ. Stud. in Math.* 71(3), 235–261.
- LEIBNIZ, G. W. (1983). *Oeuvre concernant le Calcul Infinitésimal*. Paris: A. Blanchard.
- MOTA, C. (2008). *O conceito geométrico de reta tangente: um percurso milenar*. (Unpublished MSc. thesis). Escola de Ciências, Universidade do Minho, Portugal.
- MOTA, C. & RALHA, M. E. (2011). Matemática em Portugal: marcos da história do ensino e do ensino da história. In J. Matos & M. Saraiva (Ed) *Actas do I Congresso Ibero-Americano de História da Educação Matemática*, p. 388-400. Lisboa: UIED.
- NEVES et al (2012). *Matemática A 12º Ano*. Porto: Porto Editora.
- ROBERVAL, G. P. (1736). *Ouvrages de mathematique*. Amesterdam: Chez P. Mortier.
- TZANAKIS, C: & THOMAIDIS, Y. (2011). Classifying the arguments & methodological schemes for integrating history in mathematics education. In M. Pytlak, T. Rowland & E. Swoboda (Ed), *Proceedings of the Seventh Congress of the European Society for Research in Mathematics Education*, p. 1650-1660. Rzeszow: Univeristy of Rzeszow.
- SIU, M. K. (2006). No, I don't use history of mathematics in my class. Why? In F. Furinghetti et al (Ed), *Proceedings of HPM2004 & ESU4*, p. 268-277, Uppsala: Uppsala Universitet.