REPLACING COUNTING STRATEGIES: CHILDREN'S CONSTRUCTS WORKING ON NUMBER SEQUENCES

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In mathematics didactics, consensus widely prevails on the importance of using patterns and structures for effective and flexible computation. It is equally known that children who are using counting strategies when solving problems do not perceive relations between numbers and operations. Because of this, the central objective is to replace counting strategies by realizing, recognizing and using structures. Within the context of the project ZebrA (Zusammenhänge erkennen und besprechen – Rechnen ohne Abzählen[1]) different lessons were developed to encourage children to use different interpretations of patterns and structures instead of counting. In this paper, the results of the video-based qualitative analysis of teaching/learning situations in the field of number sequences are discussed.

INTRODUCTION

Counting as computational strategy

Several studies have shown that young children are able to solve simple arithmetic problems using counting strategies (Krajewski & Schneider, 2009). Therefore, on the one hand, procedural knowledge of counting is needed. Fuson (1987) describes five levels in counting development beginning with string level, where the children interpret the number sequence as whole like a poem, followed by the levels unbreakable list, breakable chain, numerical chain and on highest level the bidirectional chain. Now it is possible for the children to count in stages forward and backward form a lower to a higher boundary. On the other hand, conceptual knowledge is necessary, like understanding the cardinal principle. During and in addition to the development of counting competency, an understanding of cardinals is developing and has to be developed (Desote, Ceulemann, Roeyers & Huylebroek, 2009). The approach to numbers by counting has to be combined with an approach to an understanding of quantities. As a result of both competences, children are able to see relationships between numerical quantities and between numbers in the numerical sequence (Krajewski & Schneider, 2009). The awareness of structural relations can be used for solving an addition or subtraction task, for example children calculate 8 + 6 no longer counting six steps on from eight but rather decomposing 6 in 2 and 4 and compute like 8 + 6 = 8 + 2 + 4 = 14. A central aim in mathematics education must be to develop these structural relations in the numerical sequence and between quantities and use them in problem solving.

[1] Recognizing and speaking about connections – calculating without counting out

But mathematical understanding does not develop in every case in the way described above. There are children who do not build up an awareness of numbers as quantities and relations. They get caught in the interpretation of numbers as ordinals and do not use structural relations between numbers when solving problems. In an actual study (Gaidoschik, 2010), a central result is that changing the strategy from deduction to the use of memorized facts takes place significantly more often by first-graders than changing from a counting strategy to memorized facts. Gaidoschik (2010) supposes a reason in the facility of counting strategies which could lead to a persistence of counting on. In the number range up to 10 or 20, counting strategies can be used successfully instead of efficient computational strategies. But counting computation is not a strategy that works in higher number ranges. Furthermore, it often comes along with a mechanical, non-reflected procedure as well as an isolated problem solving. There is a risk that the missing insights develop into comprehensive problems in mathematics education. Children with learning disabilities in mathematics are often using persistent counting strategies as their main computational strategy (Moser Opitz, 2007).

Whereas the importance of an awareness of mathematical structures for non-counting computation is stressed and can be caused in different ways, little is known about the way in which children with mathematical difficulties realize number patterns and structures. The primary question is if and how an explicit focusing on these structures in mathematics education can motivate a replacement of counting strategies. The replacement is necessary because non-counting calculation works only when using structures. Interventions focusing on the fostering of structures have led children with difficulties in mathematics to better results in standardized tests (Dowker, 2001; Kroesbergen & van Luit, 2003). But these studies do not answer the questions as to how the children realize structures and which steps are important for them to do so. Empirical findings are missing how children who are using persistent counting as their main strategies interpret numbers, operations and how they develop a structural focus on arithmetical patterns.

Interpretation of structures as a constructive and social process

As concepts of mathematics, structures and patterns are abstract and not visible. They must be constructed individually when sighting signs (Steinbring, 2005). Numbers, operation-signs, representations can be taken as signs into consideration and also as arithmetic patterns. All signs have to be interpreted. Interpretation is an active process that each child has to perform on its own, although the community in a class is important. Studies focusing the development of new mathematical knowledge of children emphasize the relevance of interactive settings (Steinbring, 2005). The new knowledge could be built up in situations where children reflect their own perception and relate it to the perception of others. As such, the students participate in the mathematical practices in the classroom, create interpretations and negotiate meanings or resolve conflicts (Cobb, Wood & Yackel, 1991). But not every cooperative and interactive act leads to new knowledge. On the one hand, the given tasks and the cooperative discourses about them seem to have great influence, on the other hand the suggestions of the teacher seem to play a central role. Directed suggestions, interventions or instructions for cooperative organized learning situations can cause communication between children, which can lead to the construction of new mathematical knowledge (Nührenbörger & Steinbring, 2009). The relevance of the teacher's influence in reorganization and interpretation of mathematical relations is shown in a qualitative study (Steinbring, 2005) and points out how important discussions among the students and the teacher can be.

PRESENT STUDY

The difficulties of children using counting as their main computational strategy can be reduced to arithmetic contents of the first two years at school (Moser Opitz, 2007). In fostering children, it is important to not only revise the contents, because an education without results becomes not more successful if it is revised in the same way (Lorenz, 2003). It must be examined if cooperative learning leads to an enhancement of individual interpretations – especially to a (more) structured focusing view of the children with persistent counting strategies.

The present study is a part of the project ZebrA. In the project, learning environments with cooperative elements for second grade in primary school or for fourth grade in special education schools have been developed, field-tested and evaluated (Häsel-Weide, Nührenbörger, Moser Opitz & Wittich, resp. 2013). Twenty units have been constructed to support children in replacing persistent counting strategies. The learning environments focus on understanding, demonstrating and imagination numbers and operations as well as the relations between them. All children of the class are taking part in the lessons. The tasks permit a fundamental awareness of mathematical structures and at the same time a deeper understanding of structures. The engaged material is sophisticated and allows learning at different stages of understanding. To initiate various interpretations, the students are working together in pairs. Each child who uses counting as its main computational strategy works with a partner who uses other strategies. The methodical design of the learning environments encourages them to exchange views on the given tasks. The tasks are given in a discursive way, so that they cause different interpretations to the end that an awareness of structures is stimulated and interpretations are enlarged.

The study was realised from September to December 2010. The teachers of the participating classes gave the lessons. They had taken part two times in an advanced further training, where they became confident with the concept of ZebrA. The ZebrA Project is accompanied by two empirical studies which allow focusing the replacement of persistent counting strategies from different empirical points of view. Whereas the quantitative study researches the effects of cooperative fostering (Wittich, Nührenbörger & Moser Opitz, 2010), the study presented in this paper focused on the interpretations of children dealing with the problems and discussing with the partner. The aim of this study is to identify and describe interpretations of children with mathematical difficulties and their development during fostering: In which situations are children able to consider and use mathematical structures? Which relations between numbers and tasks are realized in which way? In what way are the interpretations affected by the discourse with the partner or the teacher?

In order to identify children using counting strategies as persistent computation strategies as well as children using non-counting strategies the data of the quantitative study was used (results of different tests, rating of the teacher). Five children and their partner – belonging to four different classes and three schools - were chosen for the qualitative study. Their work was video-graphed in ten lessons of the ZebrA-project. Corresponding transcripts were interpreted by a group of researches (Krummheuer & Naujok, 1999). The analysis has been compared in an interactive way with empirical findings of other studies and theoretical approaches, with the result, that new insights about the interpretations of children with mathematical difficulties could be constructed. This procedure allows for the development of new theoretical elements analyzing individual cases. Concerning the present study, these could match with a characterization of typical interpretation of children using persistent counting strategies.

ANALYSIS OF AN EPISODE

The procedures and interpretations of the student Kolja (fourth grader of a special education school) working on the learning environment “number sequences” are analyzed exemplarily. First, the content of the environment is explained, followed by the presentation of the documents, procedures and interpretations noticed in the discourse with his partner Medima.

Content of the learning environment: number sequences

In this lesson, the students work on number stripes, which correspond to arithmetic sequences (Fig. 1). The sequence reflects on ordinal and relational interpretations of numbers by counting development described by Fuson (1987). Some of the stripes initiate counting in steps of one, others counting in steps of two or ten. The starting number is in isolated cases one, for the rest a number different from one. Some sequences allow counting on, some require counting back. The number sequences are given to the students on stripes and their first task is to fill them. Different stripes are given to them and each student can choose those he would like to work on. In a second step, the students sort the stripes focusing on the relations between them (such as distance between numbers of a sequence, starting number, multiplication of number or constant difference between sequences). Thereby, the attention should be changed from counting activities towards the realizing of mathematical structures. Afterwards, the students are asked to find other compatible number sequences and note them on free stripes. These self-productions could illustrate the insight of the relations on the one hand, and on the other hand, they enable children to work on their individual level of understanding.

Kolja completes number sequences

Kolja, a student using persistent counting as his main computational strategy, completes the sequences as shown in figure 1[2].

10	**20**	**30**	40	50	60	70	80
1	**2**	**3**	4	5	6	7	8
6	7	8	9	10	**11**	**13**	**15**
12	**14**	**16**	18	20	22	24	26
1	2	3	4	5	**6**	**7**	**8**

Figure 1: Reconstructed number sequences filled by Kolja

The documents show that Kolja seems to be able to find sequences with the difference of one. Equally, it seems to be no problem for him to count on in steps of ten. Sequences with the difference of two, which can be found by counting on are filled correctly, too. Only the number sequence which requires counting back in steps of two seems to be difficult for him. Using the document only, it is not possible to point out if Kolja does not realize the mathematical structure in the given number sequence or if he is not able to keep the distance when counting back. But it is striking that the wrong sequence corresponds to the highest level of the counting development described by Fuson (1987). Considering Kolja's approach in the videotape, it could be seen that Kolja counts on when he finds the numbers in the sequence _, _, _, _, _, 6, 7, 8. Starting with one, he counts on and controls if the sequence fits, as it does in that case. Similarly, Kolja finds the sequence 6, 7, 8, 9, 10, **11, 13, 15**. Two times he seems to count on tapping with a pen from left to right on the free fields before he then filled the numbers starting with 10. This may suggest that Kolja has problems in counting back generally, as pointed out by Moser Opitz (2007) to be typical of children with mathematical difficulties.

Kolja and Medima sort the stripes

After completing the stripes the teacher tells Kolja and Medima that they now should sort the stripes. She asks if the stripes fit together.

1 Kolja: Yes, that (*points to the number stripe that is located at the very bottom in front of him*)

[2] The numbers in bold print were given; the other numbers were notated by Kolja. The stripes are present in chronological order.

1	2	3	4	5	**6**	**7**	**8**

fits to that (points to the number stripe that is located as the second from the top in front of him)

1	**2**	**3**	4	5	6	7	8

2	Medima	Sorry? I didn’t understand.
3	Teacher	Put together in order, these sequences of numbers, ok?
4	Kolja	One, one, two, two, three, three, four, four, five, five, six, six, seven, seven, eight, eight
5	Medima	Oh, yeah
6	Teacher	Why do these fit?
7	Medima	Because they are the same (*pointing alternatingly to the numbers of the upper and lower number stripe*).

Kolja’s first idea to sort the number sequences is to match identical stripes. He explains this in reading out the numbers in pairs (4). Medima seems to understand immediately (5) but the teacher asks for a further explanation (6), which is given by Medima. She explains the matching with the equality of the numbers and underlines her statement in pointing to the numbers by turns.

In the further course of working, the children try to sort the other stripes in the same way, but there are no more identical sequences. Afterwards, they attempt to find matching sequences with the same starting number. Because they regard the distance between the numbers as a second factor at the same time, they do not find matching stripes. The consideration of both factors engender that they are searching for identical sequences again. To the children it is, however, not clear that considering both aspects is already the same as looking for identical stripes. They try to fit the stripes by shifting (Figure 2). But they do not find a position which allows a pair-wise order for a couple of numbers and then they give up these tries.

14	**15**	**16**	17	18	19	20	21

14	**16**	**18**	20	22	4	26	28

Figure 2: Shifted stripes

The relations considered by the children seem to be guided by the appearance of the number sequences. Exact congruence of stripes seems to be critical. They are comparing the numbers one after another with each other and are looking for identity. Matching relations like same distance between numbers or sequence initiate counting on or counting back are not discussed so far. This corresponds to other empirical findings of the ZebrA Project (Häsel-Weide, resp. 2013). When students compare an analogue, operative series of subtraction tasks they often look first for identities and stay on the surface in doing so, too. As this way to sort does not lead to results, Medima suggests a new possibility to order. She matches number sequences with the difference 10 between each position to another.

8 Medima: Found another one (*pulls two stripes over to her).*

2	4	6	8	10	**12**	**14**	**16**

12	**14**	**16**	18	20	22	24	26

9 Kolja: Yay

10 Medima: (laughs) You see? Eight (*points to the "12" on the bottom number stripe*) oh, twelve (*points once again to the "12"*), two (*points to the "2" on the top number stripe*), fourteen (*points to the "14" on the bottom number stripe*), four and so on (*moves the pair of number stripes below the one she had previously found).*

11 Kolja: (*moves two number stripes that were located in front of him together*) Look. Six, ten, seven, twenty, eight, thirty (*points each time to the number mentioned on the number stripes that are located right underneath one another*) (*points to the "9" of the top number stripe*) (.) doesn't fit

6	7	8	9	10	**11**	**13**	**15**

10	**20**	**30**	40	50	60	70	80

Medima explains to Kolja why the sequences match, in spite of the acknowledgement of Kolja before (9). In her explanation, she names the numbers pair-wise (10). She does not give a general description or an argument as she did in the dialogue with the teacher. Kolja seems to adopt the idea and tries to realize it with the residual stripes. He takes two of them and compares the numbers pair-wise, responding to their positions in the number sequence. After he has looked at three pairs, he decides that the sequences do not fit. He does not give a reason either, but reads the numbers aloud. Possibly the children check if there is a phonetic matching when reading the numbers. It is not clear if the students consider structural relations between the stripes at that time or if they confine themselves to phenomena which can be realized at the surface such as same digits at the unit position or same sound at the beginning of the numeral.

In the ongoing partner work, the children are asked to find matching sequences in free stripes. Kolja produces identical stripes, copying existing sequences. At the same time, Medima goes on with her idea to increase the numbers. Thereby she seems to mix different techniques to construct matching sequences. The single digit numbers are decupled by appending a zero, while the two digit numbers are increased by ten, changing the ten-position by one. The different chosen techniques show that Medima considers relations between pairs of numbers (3, 30 and 11, 21) more than relations between the number sequences. The different distances between the numbers do not strike her. Once all free stripes are filled out and sorted the children are approaching the teacher.

12 Teacher: I think you are so quick, you could probably think of a lot more that fit to them

…

13 Kolja What is with 100, 200, 300, 400, 500, 600, 700, 800, nine, ten

14 Teacher That is great. Where would those fit? To which that you put in order there? When you write down 100, 200, 300, 400

15 Kolja (*points to the first box of the empty stripe that is located right in front of him)*

16 Medima (*grabs the number stripe depicted below, laughs and displays it)*

10	**20**	**30**	40	50	60	70	80

17 Kolja (*fills out the empty stripe*) Yes.

100	200	300	400	500	600	700	800

…

18 Teacher Great. And now you could think of which other would fit with it. (.) How they then (*incomprehensible*).

…

19 Medima Do you already have that? Ok. (*grabs two empty number stripes that the T left on the table*) Then we have to put'em together like that (*pushes her stripe briefly underneath Kolja's stripe, then pulls it towards herself again*) But what should we then write?

20 Kolja (*pulls his stripe a little towards himself and looks at it*). Infinity, two infinity, three infinity, four infinity, five

In this episode, Kolja seems to modify Medima's idea of multiplication. Kolja's words (13) indicate that he wants to construct a sequence with big numbers and begins counting in steps starting with hundred. The question is whether Kolja realizes a relation between the existing sequences and the new ones and whether he is aware of the multiplicative relations between his sequence and an existing one. At the time when he constructs the sequence, it seems that he refers to a counting context and does not focus on relations to an existing number sequence. Medima seems to understand his idea immediately and detects a stripe that matches her idea of decupling (16). She combines Kolja's suggestion with the activity of sorting stripes. In this way, a relation between both sequences is constructed.

Both children fill a free stripe with Kolja's sequence and sort it to an existing sequence of tens. Afterwards, they ask themselves how to go on (19). Medima takes two new free stripes for each of them. She asks Kolja for an idea and he finds a new number sequence with considerably larger numbers (20). Again, it is not sure whether this sequence refers to a counting context or shows the idea "starting with a power of ten and going on in appropriate steps leads to matching sequences". Because the children have not considered the difference between the numbers in any sequence yet, the last interpretation seems perhaps too optimistic. But relations within a sequence starting with a decimal power may have been realized by Kolja.

INTERPRETATION AND CONCLUSION

The learning environments of the ZebrA Project are intended to help children using persistent counting as main computational strategy to realize mathematic structures. In the analyzed episode we were able to show that in the cooperative partner work an awareness of mathematical structures had happened.

The difficulties to count back in steps pointed out by Fuson (1987) and Moser Opitz (2007) could be seen in the solving process of Kolja. Nevertheless, Kolja has found a way to generate number sequences by testing the fitting of sequences that he generated while counting on. He may show an approach which can be observed in a similar way when children are looking for a previous number. They count on until the given number and in doing so they recognize the previous spoken one.

Sorting the stripes, but most of all finding further sequences, leads the children to dealing with the relations between the number sequences. Kolja succeeds in expanding his interpretation of equality; first he reconstructs structural relations and then uses them to find matching sequences on his own. In the realized relations, a general understanding is indicated. Both children exceed their common number space. Nevertheless, the realized relation of Kolja is limited to only one and that requires a long period of working with the material. This indicates that children with mathematical difficulties may need an extensive period of working with materials and several opportunities to become aware of mathematical structures. The ZebrA Project shows that it is possible to initiate children using persistent counting as the main computational approach to realized relations between structures, but an intensive fostering may be needed for them to become familiar with structures.

There are positive effects from Kolja's interpretation, resulting from the collaboration with Medima. She initiated an alternative interpretation and as a result the children take into consideration the mathematical relation of multiplication. The intended cooperation between a child using persistent counting as the main strategy and another works out in these instances. But Kolja is not a passive partner either: Sorting started with his idea to find equal stripes and he comes up with the number sequences with decimal power. Notwithstanding the good cooperation and communication of the students, the episode shows that the communication changes in the presence of the teacher. While the students among themselves are speaking mostly without descriptions and are using gesticulations to demonstrate their thoughts, they try to find arguments when they are asked to do so. It appears as if asking for a description or an explanation to explain the structures may be the job of the teacher.

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