# DIVERSITY IN MIDDLE SCHOOL MATHEMATICS TEACHERS' IDEAS ABOUT MATHEMATICAL MODELS: THE ROLE OF EDUCATIONAL BACKGROUND[1]

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*The purpose of this study was to explore the relationship between mathematics teachers' educational backgrounds and their ideas about 1) what constitutes a mathematical model of a real-world phenomenon, and 2) how models and empirical data relate. Participants were 56 United States (US) in-service mathematics teachers (grades 5-9). We analysed teachers' written responses to three open-ended questions through content analysis. Results show our participants do not hold a unitary understanding of mathematical models. Teachers with backgrounds in Mathematics Education and Science Disciplines especially stressed the usefulness of models to show general relationships, whereas those with backgrounds in Other Disciplines stressed the importance of producing exact results.*

## INTRODUCTION

Mathematics teachers across grades K-12 are increasingly required to include modelling in their teaching. In fact, mathematical modelling is one of eight practices standards proposed by the Common Core State Standards for Mathematics (2012). Considerable research has been devoted to exploring how mathematics teachers from different grade levels solve modelling problems (Blum & Borromeo Ferri, 2009; Verschaffel, De Corte, & Borghart, 1997) and conceive of the role of modelling activities in the classroom (Kaiser & Maass, 2007). However, little is known about how teachers understand what a "mathematical model" is, or about the relationships between models and real-world phenomena.

Defining modelling in the context of mathematics education is a complex task (English & Sriraman, 2010). Most researchers and policymakers agree that mathematical modelling involves using the tools of mathematics to distill key elements of a real-world situation and articulate the relationships between those elements. This distillation enables the learner (or the model creator, more generally) to further explore the situation using the tools of mathematics, with the ultimate purpose of mobilizing those findings toward accomplishing further goals in the original situational context (Lesh & Doerr, 2003). By definition, then, model creators themselves must decide what particular mathematical representations, tools, and methods are appropriate to use when modelling a given situation. What constitutes a

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[1] This study was funded by the National Science Foundation (NSF), Grant # DUE-0962863. The ideas expressed herein are those of the authors and do not necessarily reflect the ideas of the funding agency.

mathematical model, therefore, varies across situations and contexts, as well as users and audiences.

This study explores how middle school mathematics teachers think of mathematical models, considered as the external *objects* produced during the modelling process. Several definitions of the term "mathematical model" have been proposed by educational researchers. For example, Niss (1989) defines mathematical model as a combination of one or more mathematical "entities," whose relationships are chosen to represent aspects of a real-world situation. In a similar vein, Lesh and Doerr (2003) argue that models are conceptual systems expressed using external representations, serving as vital tools to construct, define, and explain other systems.

These definitions elaborate on models' representational nature, and emphasize how models embody the decisions modellers make during modelling process (Janvier, 1987). Mathematical models consist of one or more representations purposely chosen and displayed in a way that allows modellers to highlight what they identify as the most important variables and relationships of the phenomenon under study. Deciding what is and is not important constitutes, in fact, one of the main tasks to be completed in modelling activities. Thus, mathematical models themselves may substantially vary depending on the final goals, preferences, and/or biases of their creators. It is perhaps for that reason that the above definitions do not specify which particular representations should be considered as the "components" of mathematical models. In a nutshell, mathematical model is a "slippery" concept, open to numerous interpretations.

In this study, we use the variable "Educational Background" to explore the diversity of teachers' ideas about models as external objects. Existing literature has not yet focused on investigating the relationship between mathematics teachers' educational background and their ideas about mathematical models. In the field of science education, the interview study conducted by Justi and Gilbert (2003) explored the "notions of model" held by 39 science teachers with different disciplinary backgrounds – holding degrees in Chemistry, Physics, Biology, or Primary Teaching Certificate. The ideas of these science teachers tended to differ according to their educational backgrounds. For example, most teachers holding a Primary Teaching Certificate strongly subscribed to everyday views of the notion of model, according to which a model is a reproduction of something or a standard to be followed. Teachers with a degree in Biology expressed similar ideas, although they referred to broader variety of *uses* of models. Finally, teachers with a background in Physics or Chemistry discussed the notion of model in more comprehensive ways, consistent to perspectives currently held by scientists and philosophers of science. Moreover, they emphasized the usefulness of models for making predictions.

## PURPOSE AND JUSTIFICATION

The purpose of this study is to provide evidence that middle school mathematics teachers with different educational backgrounds tend to have diverse sets of ideas regarding 1) what constitutes a mathematical model of a real-world phenomenon, and 2) the relationships between models and empirical data. Drawing on the assumption that teachers' ideas about mathematical content are crucial mediators for the way they teach such content to their students (Sánchez & Linares, 2003), the evidence presented in this paper can be taken as an indicator that teachers with different disciplinary backgrounds might be addressing the teaching of models and modelling activities in very different ways. A better awareness of this diversity of ideas can inform teacher educators as they design programs to prepare mathematics teachers (both pre- and in-service) for an increasingly modelling-focused curriculum. Our study is also relevant for cognitive researchers interested in teacher thinking, as it shed light on the role that educational background plays in teachers' ideas about subject matter.

## METHOD

### General Curriculum and Participants

Data for this study were gathered in the context of a professional development program carried out in the Northeast region of the US. The program was composed of three graduate level on-line courses: *Representations* (Course 1), *Transformations* (Course 2), and *Invariance and Change* (Course 3).

Participants were 56 grade 5 to 9 mathematics teachers from nine school districts. There were 49 female teachers and 7 male teachers, ranging from 26 to 63 years of age. When data were collected, their professional experience as mathematics teachers ranged from 2 months to 28 years. The teachers had a variety of educational backgrounds, which we grouped into four categories:

- Mathematics: when the teachers earned their bachelor's or master's degree in mathematics (13 teachers);
- Mathematics Education: when teachers' bachelor's or master's degree was in mathematics education and they did not hold a bachelor's or master's degree in mathematics (8 teachers);
- Science Disciplines: when teachers' bachelor's or master's degree was in any science discipline (such as physics, engineering, or chemistry) and they did not hold a degree in mathematics and/or mathematics education (8 teachers);
- Other Disciplines: when teachers' bachelor's or master's degree was in other disciplines (such as Special Education, History, English, or Literature) and they did not hold a degree in any of the above-mentioned disciplines (27 teachers).

### Modelling Problem and Target Questions

The modelling problem used (see Figure 1) presents Dolbear's Law, which expresses the relationship between the rate of chirping of the snowy tree cricket ($N$) and air temperature ($T$). A linear relationship between these two variables is proposed by the model creator. To explain Dolbear's Law, different representations of this relationship were shown to the teachers. Notice that the problem explicitly characterized the algebraic expression ($N = T - 39$) as the *model.* The list of ordered pairs, tables, and graphs were characterized as *representations*. The nine data points presented in these representations were referred to as *data*.

The teachers were asked a set of 12 open-ended questions. Several questions focused on the advantages and disadvantages of some given representations over others. Other questions focused on the similarities and differences among these representations. For this study, we analysed only those questions involving the ideas of "model" and "data," which were the following:

- Question 1: *How would you characterize the relationship between the model and the data?*
- Question 2: *Could you extract the data from the model?*
- Question 3: *Do you think the model conveys more or less information than the data? Why?*

**Figure 1: Problem used in the study**

***Famous Amos and his Cricket Thermometer: Using a Function to Model Data***

Amos Dolbear (1837-1910) […] Today Dolbear is remembered, if he's remembered at all, mainly for Dolbear's Law, which expresses the relationship between the rate of chirping of the snowy tree cricket (*oecanthus fultoni*) and the air temperature. When they congregate in large numbers, these insects chirp in unison at a rate that depends on the temperature. Here are some empirical data on the subject [J.S. Walker, *Physics*, 4th ed. (Addison-Wesley, 2010), p. 451] given in the form of ordered pairs ($T$,$N$), where $T$ is the temperature in degrees Fahrenheit and $N$ is the number of chirps in 13 seconds:

(69,28), (74,34), (60,19), (77,39), (80,45), (66,23), (71,30), (57,18), (63,22)

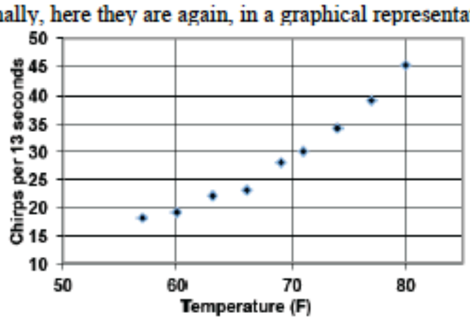
A) Here are the same data presented in tabular form:

| Temperature in degrees Fahrenheit $T$ | Chirps per 13 seconds $N$ |
|---|---|
| 69 | 28 |
| 74 | 34 |
| 60 | 19 |
| 77 | 39 |
| 80 | 45 |
| 66 | 23 |
| 71 | 30 |
| 57 | 18 |
| 63 | 22 |

B) and the same data again in tabular form, but this time in order of increasing temperature:

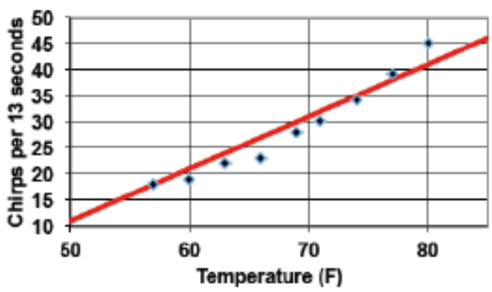
| Temperature in degrees Fahrenheit $T$ | Chirps per 13 seconds $N$ |
|---|---|
| 57 | 18 |
| 60 | 19 |
| 63 | 22 |
| 66 | 23 |
| 69 | 28 |
| 71 | 30 |
| 74 | 34 |
| 77 | 39 |
| 80 | 45 |

C) Finally, here they are again, in a graphical representation:




Based on these data, Dolbear proposed a *model* for the relationship between chirp rate and temperature, expressed in the following formula: $N = T - 39$ where, again, $N$ is the number of chirps per 13 seconds, and $T$ is the temperature in degrees Fahrenheit.

D) Here is a graphical representation of the model, plotted together with the actual data.



## RESULTS

Teachers' responses to the three target questions were analysed using sets of non-mutually exclusive categories. In this section, we first focus on the content specific to each question. Then, we analyse some additional ideas that systematically emerged throughout the three questions. Tables containing the frequencies and percentages obtained by each group are presented. Examples of characteristic responses produced by the participating teachers will be shown in our oral presentation.

### Question 1: How would you characterize the relationship between the model and the data?

Teachers with different educational backgrounds tended to refer to different representations when characterizing the notion of "mathematical model" (Table 1). Recall that all sets of categories used in this study are non-mutually exclusive in nature. This is why counts in many columns of the tables presented go over 100%.

| | EDUCATIONAL BACKGROUND | | | |
|---|---|---|---|---|
| | Mathemat N=13 | Math Ed N= 8 | Science N=8 | Other Disc N=27 |
| *The model is referred to…* | | | | |
| As an Algebraic Expression | **6 (46.1%)** | **5 (62.5%)** | 1 (12.5%) | 3 (11.1%) |
| As the Data Points | 1 (7.7%) | 1 (12.5%) | **3 (37.5%)** | **9 (33.3%)** |
| As the Line of Best Fit | **8 (61.5%)** | **4 (50%)** | **7 (87.5%)** | **17 (62.9%)** |
| Doesn't specify / Unclear | 2 (15.3%) | 0 | 0 | 0 |

**Table 1. Frequencies and percentages obtained by Question 1 analytic categories***

* Values equal or higher than a third of each subsample (33.3%) are bold. Categories are non-mutually exclusive

Teachers from all educational backgrounds frequently referred to the line of best fit as the model, especially the group Science Disciplines. The groups Mathematics and Mathematics Education tended to express that the model was the algebraic expression provided ($N = T - 39$). They referred to it using the terms "equation," "formula," and/or "function." In addition, some teachers from the groups Science Disciplines and Other Disciplines referred to the original data points as being part of the model. Interestingly, other representations provided such as the ordered pairs, the tables, and the written description of the scenario were never referred to as the model.

### Question 2: Could you extract the data from the model?

We found opposite ideas regarding the viability of extracting the original data points from the model (Table 2). Whereas the groups Mathematics and Mathematics Education provided us mostly with negative answers (i.e., the data points *cannot* be extracted from the model), the groups Science Disciplines and Other Disciplines showed more diverse views, providing both affirmative and negative responses. These findings make perfect sense if we consider that the data points are regarded as part of the model by groups Science Disciplines and Other Disciplines, but not by the groups Mathematics and Mathematics Education.

| | EDUCATIONAL BACKGROUND | | | |
|---|---|---|---|---|
| | Mathemat N=13 | Math Educat N= 8 | Science Disc N=8 | Other Disc N=27 |
| Affirmative answer | 2 (15.3%) | 1 (12.5%) | **4 (50%)** | **10 (37%)** |
| Negative answer | **11 (84.6%)** | **5 (62.5%)** | 2 (25%) | **14 (51.8%)** |
| Elements of both | 0 | 1 (12.5%) | 0 | 0 |
| Unclassifiable / Unclear | 0 | 1 (12.5%) | 2 (25%) | 3 (11.1%) |

**Table 2. Frequencies and percentages obtained by Question 2 analytic categories***

Values equal or higher than a third of each subsample (33.3%) are bold. Categories are non-mutually exclusive.

### Question 3: Do you think the model conveys more or less information than the data? Why?

This question also elicited different responses among our participating teachers. Most teachers from all backgrounds think that the model conveys more information than the data, although the justifications given present interesting differences (justifications are not presented here due to space limitations). In addition, the opposite idea (the model conveys less information) was also identified among some teachers (Table 3).

| | EDUCATIONAL BACKGROUND | | | |
|---|---|---|---|---|
| | Mathemat N=13 | Math Educat N= 8 | Science Disc N=8 | Other Disc N=27 |
| More information | **10 (76.9%)** | **5 (62.5%)** | **5 (62.5%)** | **13 (48.1%)** |
| Less information | 0 | 2 (25%) | 0 | 5 (18.5%) |
| It depends, Different, More and Less | 3 (23.1%) | 1 (12.5%) | 2 (25%) | 5 (18.5%) |
| Unclassifiable / Unclear | 0 | 0 | 1 (12.5%) | 4 (14.8%) |

**Table 3. Frequencies and percentages obtained by Question 2 analytic categories***

* Values equal or higher than a third of each subsample (33.3%) are bold. Categories are non-mutually exclusive.

### An analysis across Questions 1, 2, and 3: Exactness *vs.* Relationship

Content analysis led us identify the existence of two ideas that recurrently appeared in most teachers' responses throughout the three questions analysed here: the ideas of "Exactness" (or lack thereof) and "Relationship." This finding is interesting to us because the questions posed did not explicitly ask about these issues.

As can be seen in Table 4, most teachers from the groups Mathematics Education and Science Disciplines consistently referred to the usefulness of models to show relationships or general patterns in the data. This result was consistently found across the three questions analysed. Very few references to the idea of exactness were identified among these two groups. Teachers from the group Other Disciplines, in contrast, tended to stress the idea of exactness very often. For them, models should be able to provide exact/precise approximations to empirical data. Teachers from the group Mathematics did not show a clearly defined tendency. Interestingly, Question 3 triggered responses focused on both ideas (exactness and relationship), although specially that of relationship.

| | EXACTNESS | | | RELATIONSHIP | | |
|---|---|---|---|---|---|---|
| | Question 1 | Question 2 | Question 3 | Question 1 | Question 2 | Question 3 |
| Mathematics (N = 13) | 5 (38.4%) | 5 (38.4%) | 6 (46.1%) | 5 (38.4%) | 3 (23%) | **8 (61.5%)** |
| Math Educat (N = 8) | 0 | 2 (25%) | 3 (37.5%) | **6 (75%)** | **5 (62.5%)** | **5 (62.5%)** |
| Science Disci (N = 8) | 1 (25%) | 3 (37.5%) | 2 (25%) | **7 (87.5%)** | **6 (75%)** | **8 (100%)** |
| Other Discipli (N = 27) | **17 (62.9%)** | **16 (59.2%)** | **20 (74%)** | 8 (29.6%) | 11 (40.7%) | 12 (44.4%) |

**Table 4. Exactness vs. Relationship: Frequencies and relative percentages***

* Values equal or higher than higher than a half of each subsample (50%) are highlighted. Categories are non-mutually exclusive.

## DISCUSSION AND CONCLUSIONS

Researchers have proposed that mathematical models consist of one or several mathematical entities purposely chosen and displayed to predict and/or explain phenomena (e.g., Lesh & Doerr, 2003; Niss, 1989). Consistent with that idea, the first conclusion of this study is that middle school mathematics teachers also understand and think of mathematical models as external tools of representational nature (Janvier, 1987). In the context of “The Cricket Problem,” most teachers cited several external representations as constituting the model, including the line of best fit, algebraic expressions (equation, formula, and/or function), and to a lesser extent, the raw data points. Teachers associated these representations with the model even though the text of the problem explicitly defined the algebraic expression as the model and characterized the others as representations. Other external representations available – such as the list of ordered pairs, the written description of the scenario, the unordered table, and the ordered table – were never explicitly referred to as part of the model, even though some of them contained precisely the same information as the data points.

Our second conclusion entails two ideas: a) middle school mathematics teachers do not hold unitary understandings either of what a mathematical model *is* or of what such a model is *for*; and b) teachers with different educational backgrounds understand mathematical models in systematically different ways (Justi & Gilbert, 2003). This study illustrates the extent to which the notion “mathematical model” can be interpreted differently by different audiences. Even though all teachers were provided with the same materials, they spontaneously focused their attention on different representations and analysed the situation using different criteria. While teachers from all backgrounds tended to include the line of best fit in the model, teachers with backgrounds in Mathematics and Mathematics Education were more likely than those in the other two groups to mention algebraic expressions, and to state that the data could not be extracted from the model (those with Mathematics degrees were most emphatic on this point). In contrast, those with backgrounds in Science or Other Disciplines were more likely to consider the data points themselves as part of the model, and thus to consider it possible, if perhaps difficult, to extract the data from the model.

A somewhat different pattern emerges, however, when the teachers' reasoning is considered more closely, with attention to the recurring themes of "exactness" and "relationship." Teachers with backgrounds in Mathematics Education and in Science Disciplines strongly stressed the important role of the model in illustrating or clarifying the general relationship between the two quantities. For them, models are powerful tools to show general patterns in the data and to be able to generalize beyond the particular set of data at hand to the natural phenomenon under study. Teachers from a Mathematics background tended to take a more formal approach, viewing the data and model as mathematical objects to be compared, with less concern for the real-world phenomenon being described. Teachers with backgrounds in Other Disciplines were most concerned with exactness – whether the model could precisely reproduce the original data.

An educational background in Mathematics seems to be associated with understanding models as an "idealization" of the data, as abstract representational tools whose main work is to *predict*. In addition, teachers with this background are concerned with formal aspects of models. For example, they frequently expressed the importance of clarifying the domain and range of the function, and referred to the tensions between mathematics and the natural world (e.g., crickets would die under extreme temperature conditions). Finally, these teachers focused on the question of how the model and data are related as "abstract entities" (Kaiser & Maass, 2007), with less reference to what the user might want the models for.

Teachers with educational backgrounds in Mathematics Education and Science Disciplines demonstrated some similar ideas to each other. They focused on what the model is good for, as well as what the user can do with models. In addition, they described models as powerful tools not only to *predict* but also to convey the relationship between variables and to help us see patterns and generalize (Justi & Gilbert, 2003). Exactness of models is not an important issue for these teachers, as it is for the Other Disciplines group. Teachers with a Mathematics Education background extensively referred to the advantages of having a model (e.g., to visualize patterns, estimate unknown data, see trends in the data). However, they did not describe in detail the specific conceptual information of the problem at hand, as teachers from Science Disciplines did. Indeed, having a background in Science Disciplines is associated with focusing on the specific characteristics of the model (specific kind of functional relationship, strength of the relationship, presence of outliers, etc.). Moreover, it was primarily the Science Disciplines group who tended to see the model presented as just one of many possible models, and who discussed other possibilities (i.e., an exponential model).

Formal education received by teachers from the Other Disciplines group was focused neither on mathematics nor science content knowledge. The preoccupation of these teachers with the exactness of the model suggests that they might conceive of mathematics as an abstract, authoritarian discipline (Kaiser & Maass, 2007). Like many college students, teachers from the Other Disciplines group tend to view models as either exactly right or else completely arbitrary, in which case the choice of a model becomes entirely subjective. The characterization of the "degree of exactness" of models along a *spectrum* is a more sophisticated idea than just dividing them into "exact" and "non-exact" (or "right" and "wrong"), as teachers in the Other Disciplines group did. Similar to the Mathematics Education group, these teachers did not pay much attention to the conceptual/contextual information of the modelling scenario at hand. In contrast, they tended to refer to the model in the abstract (Verschaffel et al., 1997).

## EDUCATIONAL IMPLICATIONS

Given the diversity of potential educational backgrounds among middle school mathematics teachers, it is crucial to develop a better understanding of the ways in which these diverse backgrounds might influence their ideas about mathematical models, and subsequently their teaching of modelling. The findings of this study can inform the design of units on mathematical modelling for both pre-service and in-service mathematics teachers. For example, it would be enriching for teachers with backgrounds in Mathematics and Mathematics Education to deal with situations of exploration and analysis of the different every-day constraints that might affect mathematical models. Similarly, teachers with backgrounds in Other Disciplines would benefit from experiences in which the exactness of models is not an essential issue. This would allow them to explore the advantages of visualizing general trends in the data. More generally, the evidence presented here shows that there is room for all teachers – regardless their educational background – to expand the range of representations they consider as, or include in, mathematical models, and the goals and purposes of generating, analysing, and evaluating such models. Our findings further suggest that one way to do this might be to encourage teachers with different backgrounds to collaboratively engage in modelling activities, in order to better understand the role of perspective, available tools and skills, and sense-making play in modelling activity.

## LIMITATIONS AND DIRECTIONS FOR FURTHER RESEARCH

This study is clearly exploratory and suffers from a number of limitations. First, the evidence comes from a single source of data —i.e., written responses to open-ended questions. The present analysis constitutes the first step in our research agenda on middle school mathematics teachers' modelling ideas and approaches. Other data sources should be included to validate our claims. This is precisely our next goal. We are currently in the process of interviewing a subset of the teachers who participated in this study. Among the goals of our interviews is to clarify some of the findings presented here. Another limitation is that the sample of participating teachers was uneven regarding the four educational background groups, and some of the groups were too small for robust statistical analyses. This imbalance roughly reflects the current backgrounds among middle school mathematics teachers in the US, where teachers with a Mathematics or Mathematics Education background are outnumbered by teachers with backgrounds in Other Disciplines. The fact that this study was conducted in the context of a professional development program did not allow us to select the sample having the educational background criterion in mind. Therefore, further studies should be conducted to determine whether the differences identified here are also observed in other samples of mathematics teachers. It would be also necessary to study mathematics teachers' responses in other types of modelling situations (e.g., probabilistic simulation situations, theory-driven models).

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